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FURTHER DEVELOPMENT OF CHEBYSHEV TYPE INEQUALITIES FOR SUGENO INTEGRALS AND T-(S-)EVALUATORS

Hamzeh Agahi, Radko Mesiar and Yao Ouyang

In this paper further development of Chebyshev type inequalities for Sugeno integrals based on an aggregation function H and a scale transformation φ is given. Consequences for T-(S-)evaluators are established.

Keywords: Sugeno integral, fuzzy measure, comonotone functions, Chebyshev's inequality, t-norm; t-conorm, T-(S-)evaluators

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1. INTRODUCTION

In 1974, Sugeno [25] initiated researches on fuzzy measures and fuzzy integrals. Fuzzy measure is a generalization of the notion of measure in mathematical analysis (see, e. g., [18, 26]). Sugeno integral is analogous to Lebesgue integral which has been studied by many authors, including Pap [18], Ralescu and Adams [19], Román–Flores et al. [6, 20, 21, 22, 23] and, Wang and Klir [26], among others. The difference between Sugeno integral and Lebesgue integral is that addition and multiplication in the definition of Lebesgue integral are replaced respectively by the operations max and min when Sugeno integral is considered. Román–Flores et al. [6, 20, 21, 22, 23], started the studies of inequalities for Sugeno integral, and then followed by the authors [1, 2, 11, 12, 13]. In [11], Mesiar and Ouyang proved the following Chebyshev type inequalities for Sugeno integrals:

Theorem 1.1. Let $f, g \in \mathcal{F}_+(X)$ and μ be an arbitrary fuzzy measure such that $(S)\int_A f\,\mathrm{d}\mu$ and $(S)\int_A g\,\mathrm{d}\mu$ are finite. Let $\star\colon [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If f, g are comonotone, then the inequality

$$(S)\int_A f \star g\,\mathrm{d}\mu \geq \left((S)\int_A f\,\mathrm{d}\mu\right) \star \left((S)\int_A g\,\mathrm{d}\mu\right) \tag{1}$$

holds.

It is known that

$$(S)\int_A f\star g\,\mathrm{d}\mu \le \left((S)\int_A f\,\mathrm{d}\mu\right)\star\left((S)\int_A g\,\mathrm{d}\mu\right) \tag{2}$$

where f, g are comonotone functions whenever $\star \ge \max$ (for a similar result, see [15]), it is of great interest to determine the operator $\star$ such that

$$(S)\int_A f\star g\,\mathrm{d}\mu = \left((S)\int_A f\,\mathrm{d}\mu\right)\star\left((S)\int_A g\,\mathrm{d}\mu\right) \tag{3}$$

holds for any comonotone functions f, g, and for any fuzzy measure μ and any measurable set A. Ouyang et al. [16, 17] proved that there are only 18 operators such that (3) holds, including the four well-known operators: minimum, maximum, PF (called the first projection, PF for short, if $x\star y = x$ for each pair (x, y)) and PL (called the last projection, PL for short, if $x\star y = y$ for each pair (x, y)).

Recently, Agahi and Yaghoobi [1] proved a Minkowski type inequality for monotone functions and arbitrary fuzzy measure-based Sugeno integrals on real line, and then Agahi et al. [2] further generalized it to comonotone functions and arbitrary fuzzy measure-base Sugeno integrals on an arbitrary measurable space. For (1), Agahi, Mesiar and Ouyang [3, 14] gave some strengthened versions and extensions. In particular, in [14], Ouyang *et al* presented a generalization of Chebyshev inequality for Sugeno integral given in Theorem 1.2) below.

Theorem 1.2. Let $f, g \in \mathcal{F}_+(X)$ and μ be an arbitrary fuzzy measure such that both $(S)\int_A f^s\,\mathrm{d}\mu$ and $(S)\int_A g^s\,\mathrm{d}\mu$ are finite. And let $\star\colon [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If f, g are comonotone, then the inequality

$$\left((S)\int_A (f\star g)^s\,\mathrm{d}\mu\right)^{\frac{1}{s}} \ge \left((S)\int_A f^s\,\mathrm{d}\mu\right)^{\frac{1}{s}}\star\left((S)\int_A g^s\,\mathrm{d}\mu\right)^{\frac{1}{s}} \tag{4}$$

holds for all $0 < s < \infty$.

The aim of this paper is a deep generalization of some of the above results based on an aggregation function H and a scale transformation φ.

The paper is arranged as follows. For convenience of the reader, in the next section, we review some preliminaries and summarization of some previous known results. In Section 3, we construct further development of Chebyshev type inequalities for Sugeno integrals and relate them to T-evaluators and S-evaluators. Finally, a conclusion is given.

2. PRELIMINARIES

In this section, we are going to review some well known results from the theory of fuzzy measures, Sugeno integrals and T-(S-)evaluators. For details, we refer to [9, 19, 25, 26] and [5]. For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper (see [2, 16, 17]).

As usual we denote by $\mathbb{R}$ the set of real numbers. Let X be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of X and m be the Lebesgue measure on $\mathbb{R}$. Let $\mathbf{N}$ denote the set of all positive integers and $\overline{\mathbb{R}_+}$ denote $[0, +\infty]$. Throughout this paper, we fix the measurable space $(X, \mathcal{F})$, and all considered subsets are supposed to belong to $\mathcal{F}$.

Definition 2.1. (Ralescu and Adams [19]) A set function $\mu : \mathcal{F} \to \overline{\mathbb{R}_+}$ is called a fuzzy measure if the following properties are satisfied:

(FM1) $\mu(\emptyset) = 0$;

(FM2) $A \subset B$ implies $\mu(A) \leq \mu(B)$;

(FM3) $A_1 \subset A_2 \subset \cdots$ implies $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$;

(FM4) $A_1 \supset A_2 \supset \cdots$, and $\mu(A_1) < +\infty$ imply $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

When μ is a fuzzy measure, the triple $(X, \mathcal{F}, \mu)$ then is called a fuzzy measure space.

Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, by $\mathcal{F}_+(X)$ we denote the set of all nonnegative measurable functions $f : X \longrightarrow [0, \infty)$ with respect to $\mathcal{F}$. In what follows, all considered functions belong to $\mathcal{F}_+(X)$. Let f be a nonnegative real-valued function defined on X, we will denote the set $\{x \in X | f(x) \geq \alpha\}$ by F_α for $\alpha \geq 0$. Clearly, F_α is nonincreasing with respect to α, i. e., $\alpha \leq \beta$ implies $F_\alpha \supseteq F_\beta$. Moreover, for any fixed k in $(0, \infty)$ denote by $\mathcal{F}_k(X)$ the set of all measurable functions $f : X \longrightarrow [0, k]$. Observe that the system $(\mathcal{F}_k(X))$ is strictly increasing and $\bigcup \mathcal{F}_k(X) \subsetneqq \mathcal{F}_+(X)$.

Definition 2.2. (Pap [18], Sugeno [25], Wang and Klir [26]) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $A \in \mathcal{F}$, the Sugeno integral of f on A, with respect to the fuzzy measure μ, is defined as

$$(S) \int_A f \, \mathrm{d}\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)).$$

When $A = X$, then

$$(S) \int_X f \, \mathrm{d}\mu = (S) \int f \, \mathrm{d}\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(F_\alpha)).$$

It is well known that Sugeno integral is a type of nonlinear integral [10]. I. e., for general case,

$$(S) \int (af + bg) \, \mathrm{d}\mu = a(S) \int f \, \mathrm{d}\mu + b(S) \int g \, \mathrm{d}\mu$$

does not hold. Some basic properties of Sugeno integral are summarized in [18, 26], we cite some of them in the next theorem.

Theorem 2.3. (Pap [18], Wang and Klir [26]) Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, then

(i) $\mu(A \cap F_\alpha) \geq \alpha \Longrightarrow (S)\int_A f\,\mathrm{d}\mu \geq \alpha$;

(ii) $\mu(A \cap F_\alpha) \leq \alpha \Longrightarrow (S)\int_A f\,\mathrm{d}\mu \leq \alpha$;

(iii) $(S)\int_A f\,\mathrm{d}\mu < \alpha \Longleftrightarrow$ there exists $\gamma < \alpha$ such that $\mu(A \cap F_\gamma) < \alpha$;

(iv) $(S)\int_A f\,\mathrm{d}\mu > \alpha \Longleftrightarrow$ there exists $\gamma > \alpha$ such that $\mu(A \cap F_\gamma) > \alpha$;

(v) If $\mu(A) < \infty$, then $\mu(A \cap F_\alpha) \geq \alpha \Longleftrightarrow (S)\int_A f\,\mathrm{d}\mu \geq \alpha$;

(vi) If $f \leq g$, then $(S)\int f\,\mathrm{d}\mu \leq (S)\int g\,\mathrm{d}\mu$.

In [12], Ouyang and Fang proved the following result which generalized the corresponding one in [22].

Lemma 2.4. Let $f\colon [0,\infty) \to [0,\infty)$ be a nonincreasing function. If $(S)\int_0^a f\,\mathrm{d}m = p$, then

$$f(p-) \geq (S)\int_0^a f\,\mathrm{d}m = p$$

for all $a \geq 0$, where $f(p-) = \lim_{x \to p^-} f(x)$.
Moreover, if $p < a$ and f is continuous at p, then $f(p-) = f(p) = p$.

Notice that if f is nonincreasing, then $f(p-) \geq p$ implies $(S)\int_0^a f\,\mathrm{d}m \geq p$ for any $a \geq p$. In fact, the monotonicity of f and the fact $f(p-) \geq p$ imply that $[0,p) \subset F_p$. Thus, $m([0,a] \cap F_p) \geq m([0,a] \cap [0,p)) = m([0,p)) = p$. Now, by Theorem 2.3(i), we have $(S)\int_0^a f\,\mathrm{d}m \geq p$.

Based on Lemma 2.4, Ouyang et al. proved some Chebyshev type inequalities [13] and their united form [11] (i.e., Theorem 1.1 in this paper). Notice that when proving these theorems, the following lemma, which is derived from the transformation theorem for Sugeno integrals (see [26]), plays a fundamental role.

Lemma 2.5. Let $(S)\int_A f\,\mathrm{d}\mu = p$. Then $\forall\, r \geq p$, $(S)\int_A f\,\mathrm{d}\mu = (S)\int_0^r \mu(A \cap F_\alpha)\,\mathrm{d}m$.

In this contribution, we will prove further development of Chebyshev type inequalities for Sugeno integrals and T-(S-)evaluators of comonotone functions. Recall that two functions $f, g\colon X \to R$ are said to be comonotone if for all $(x,y) \in X^2$, $(f(x) - f(y))(g(x) - g(y)) \geq 0$. Clearly, if f and g are comonotone, then for all non-negative real numbers p, q either $F_p \subset G_q$ or $G_q \subset F_p$. Indeed, if this assertion does not hold, then there are $x \in F_p \backslash G_q$ and $y \in G_q \backslash F_p$. That is,

$$f(x) \geq p, g(x) < q \quad \text{and} \quad f(y) < p, g(y) \geq q,$$

and hence $(f(x) - f(y))(g(x) - g(y)) < 0$, contradicts with the comonotonicity of f and g. Notice that comonotone functions can be defined on any abstract space.

Now, we give the following definitions which will be used later.

Definition 2.6. (Bodjanova and Kalina [5]) For a complete lattice $(X, \leq, \perp, \top)$ with the least and the greatest elements $\perp$ and $\top$, respectively, a function $\varphi : X \to [0,1]$ is said to be an evaluator on X iff it satisfies the following properties:

(1) $\varphi(\perp) = 0, \varphi(\top) = 1$.

(2) for all $a, b \in X$, if $a \leq b$ then $\varphi(a) \leq \varphi(b)$.

Definition 2.7. (Klement et al. [9]) A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is said to be a t-norm iff it satisfies the following properties:

(i) for each $y \in [0,1]$ $T(1, y) = y$,

(ii) for all $x, y \in [0,1]$ $T(x, y) = T(y, x)$,

(iii) for all $x, y_1, y_2 \in [0,1]$ if $y_1 \leq y_2$ then $T(x, y_1) \leq T(x, y_2)$,

(iv) for all $x, y, z \in [0,1]$ $T(x, T(y, z)) = T(T(x, y), z)$.

The four basic t-norms are:

- the minimum t-norm, $T_M(x, y) = \min\{x, y\}$,
- the product t-norm, $T_P(x, y) = x.y$,
- the Łukasiewicz t-norm, $T_L(x, y) = \max\{0, x + y - 1\}$,
- the drastic product,

$$T_D(x, y) = \begin{cases} 0 & \text{if} \quad \max\{x, y\} < 1, \\ \min\{x, y\} & \text{if} \quad \max\{x, y\} = 1. \end{cases}$$

A function $S : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm [9], if there is a t-norm T such that $S(x, y) = 1 - T(1 - x, 1 - y)$. Evidently, a t-conorm S satisfies:

(i′) $S(x, 0) = S(0, x) = x$, $\forall\, x \in [0,1]$ as well as conditions (ii), (iii) and (iv). The basic t-conorms (dual of four basic t-norms) are:

- the maximum t-conorm, $S_M(x, y) = \max\{x, y\}$,
- the probabilistic sum, $S_P(x, y) = x + y - xy$,
- the Łukasiewicz t-conorm, $S_L(x, y) = \min\{1, x + y\}$,
- the drastic sum,

$$S_D(x, y) = \begin{cases} 1 & \text{if} \quad \min\{x, y\} > 0, \\ \max\{x, y\} & \text{if} \quad \min\{x, y\} = 0. \end{cases}$$

Definition 2.8. (Bodjanova and Kalina [5]) Consider a complete lattice $(X, \leq, \perp, \top)$, a t-norm T and a t-conorm S. An evaluator on X is called a T-evaluator iff for all $a, b \in X$

$$T(\varphi(a), \varphi(b)) \leq \varphi(a \wedge b),$$

and it is called an S-evaluator iff

$$S(\varphi(a), \varphi(b)) \geq \varphi(a \vee b).$$

3. MAIN RESULTS

The aim of this paper is to give the following results.

Theorem 3.1. Let $k \in (0, \infty)$ be fixed. For any continuous and non-decreasing $\varphi : [0, k] \to [0, k]$ satisfying $\varphi(x) \le x$ for all $x \in [0, k]$ and any non-decreasing n-place function $H : [0, \infty)^n \to [0, \infty)$ such that H is continuous and bounded from above by minimum and any comonotone system $f_1, f_2, \ldots, f_n$ from $\mathcal{F}_k(X)$ and any fuzzy measure μ it holds

$$\begin{aligned} & (S)\int_A H\left(\varphi(f_1), \varphi(f_2) \ldots, \varphi(f_n)\right) \mathrm{d}\mu \\ \ge \; & H\left(\varphi\left((S)\int_A f_1 \,\mathrm{d}\mu\right), \varphi\left((S)\int_A f_2 \,\mathrm{d}\mu\right) \ldots, \varphi\left((S)\int_A f_n \,\mathrm{d}\mu\right)\right). \end{aligned} \tag{5}$$

Proof. Let $(S)\int_A f_i \,\mathrm{d}\mu = p_i$, $i = 1, 2, \ldots, n$. Theorem 2.3(v) implies that

$$(S)\int_A f_i \,\mathrm{d}\mu = p_i \Longrightarrow \mu\left(A \cap \{x | f_i(x) \ge p_i\}\right) \ge p_i.$$

Then

$$\mu\left(A \cap \{x | \varphi(f_i)(x) \ge \varphi(p_i)\}\right) \ge p_i.$$

Since $\varphi(x) \le x$ for all $x \in [0, k]$ and $H : [0, \infty)^n \to [0, \infty)$ is continuous and bounded from above by minimum there holds

$$H\left(\varphi(p_1), \varphi(p_2) \ldots, \varphi(p_n)\right) \le \min\left(\varphi(p_1), \varphi(p_2), \ldots, \varphi(p_n)\right) \le \min\left(p_1, p_2 \ldots, p_n\right).$$

Therefore

$$\begin{aligned} & \mu\left(A \cap \{x | H\left(\varphi(f_1), \varphi(f_2) \ldots, \varphi(f_n)\right) \ge H\left(\varphi(p_1), \varphi(p_2) \ldots, \varphi(p_n)\right)\}\right) \\ \ge \; & \mu\left(\begin{array}{c} A \cap \{x | \varphi(f_1)(x) \ge \varphi(p_1)\} \\ \cap \{x | \varphi(f_2)(x) \ge \varphi(p_2)\} \cap \ldots \cap \{x | \varphi(f_n)(x) \ge \varphi(p_n)\} \end{array}\right) \\ = \; & \min\left(\mu\left(A \cap \{x | \varphi(f_1)(x) \ge \varphi(p_1)\}\right), \ldots, \mu\left(A \cap \{x | \varphi(f_n)(x) \ge \varphi(p_n)\}\right)\right) \\ \ge \; & \min\left(p_1, p_2, \ldots, p_n\right) \ge H\left(\varphi(p_1), \varphi(p_2), \ldots, \varphi(p_n)\right), \end{aligned}$$

and, consequently, from Theorem 2.3(i) we obtain:

$$(S)\int_A H\left(\varphi(f_1), \ldots, \varphi(f_n)\right) \mathrm{d}\mu \ge H\left(\varphi\left((S)\int_A f_1 \,\mathrm{d}\mu\right), \ldots, \varphi\left((S)\int_A f_n \,\mathrm{d}\mu\right)\right).$$

This completes the proof. □

Remark 3.2. If $H(k, \ldots, k) = k$, then the function H required in Theorem 3.1 is a conjunctive (continuous) aggregation function on $[0, k]$, compare [7]. Typical examples of such functions on $[0, 1]$ interval, i. e., if $k = 1$, are (continuous) t-norms, copulas, quasi-copulas, etc. Note also that the function φ required in Theorem 3.1 can be seen as a (contracting) transformation of the scale $[0, k]$.

Corollary 3.3. Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g \colon X \to [0,1]$ two comonotone measurable functions. If T is a continuous t-norm and φ is continuous T-evaluator on X such that $\varphi(x) \leq x$, then the inequality

$$(S) \int_A T(\varphi(f), \varphi(g)) \, \mathrm{d}\mu \geq T\left(\varphi\left((S)\int_A f \, \mathrm{d}\mu\right), \varphi\left((S)\int_A g \, \mathrm{d}\mu\right)\right) \tag{6}$$

holds for any $A \in \mathcal{F}$.

The following example shows that $\varphi(x) \leq x$ for all $x \in [0, k]$ in Theorem 3.1 is inevitable.

Example 3.4. Let $X \in [0, \frac{1}{2}]$, $f_1(x) = x$, $f_2(x) = \frac{1}{2}$, $\varphi(x) = \sqrt{x}$ and $H(x,y) = x.y$. Then

$$(S) \int (\varphi(f_1) \times \varphi(f_2)) \, \mathrm{d}\mu = 0.309\,02 \,, \varphi\left((S)\int f_1 \, \mathrm{d}\mu\right) = \frac{1}{2}$$

and

$$\varphi\left((S)\int f_2 \, \mathrm{d}\mu\right) = \sqrt{\frac{1}{2}},$$

but

$$\begin{aligned} 0.309\,02 &= (S)\int H(\varphi(f_1), \varphi(f_2)) \, \mathrm{d}\mu < H\left(\varphi\left((S)\int f_1 \, \mathrm{d}\mu\right), \varphi\left((S)\int f_2 \, \mathrm{d}\mu\right)\right) \\ &= 0.353\,55, \end{aligned}$$

which violates Theorem 3.1.

The following example shows that the comonotonicity of $f_1, f_2, \ldots, f_n$ in Theorem 3.1 cannot be omitted.

Example 3.5. Let $X = [0,1]$, $f_1(x) = \sqrt{x}$, $f_2(x) = \sqrt{1-x}$, $H(x,y) = \min\{x, y\}$, $\varphi(x) = x^2$ and the fuzzy measure μ be defined as $\mu(A) = m(A)$. Then

$$(S)\int f_1 \, \mathrm{d}\mu = (S)\int f_2 \, \mathrm{d}\mu = \bigvee_{\alpha \in [0,1]} \left[\alpha \wedge (1 - \alpha^2)\right] = 0.618\,03,$$

and

$$(S)\int (f_1^2 \wedge f_2^2) \, \mathrm{d}\mu = \bigvee_{\alpha \in [0, \frac{1}{2}]} \alpha \wedge (1 - 2\alpha) = 0.333\,33.$$

Hence

$$\begin{aligned} (S)\int (\varphi(f_1) \wedge \varphi(f_2)) \, \mathrm{d}\mu &= 0.333\,33 < 0.381\,96 \\ &= \varphi\left((S)\int f_1 \, \mathrm{d}\mu\right) \bigwedge \varphi\left((S)\int f_2 \, \mathrm{d}\mu\right), \end{aligned}$$

which violates Theorem 3.1.

The following results are easy to obtain.

Corollary 3.6. Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g\colon X \to [0,1]$ two comonotone measurable functions. Let $\star\colon [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from above by minimum. Then

$$(S)\int_A f \star g \,\mathrm{d}\mu \geq \left((S)\int_A f\,\mathrm{d}\mu\right) \star \left((S)\int_A g\,\mathrm{d}\mu\right) \tag{7}$$

for any $A \in \mathcal{F}$.

Corollary 3.7. Let $f_1, f_2, \ldots, f_n$ be such that any two of them are comonotone and be as in Corollary 3.3. Then

$$\begin{aligned}
&\left((S)\int_A \varphi\left(T\left(\ldots\left(T\left(T\left(f_1, f_2\right)\right), f_3\right)\ldots\right), f_n\right)\mathrm{d}\mu\right)\\
\geq\;& T\left(\begin{array}{c} T\left(\ldots\left(T\left(\varphi\left((S)\int_A f_1\,\mathrm{d}\mu\right)\right), \left(\varphi\left((S)\int_A f_2\,\mathrm{d}\mu\right)\right)\right), \ldots \\ \ldots, \left(\varphi\left((S)\int_A f_n\,\mathrm{d}\mu\right)\right)\end{array}\right).
\end{aligned}$$

Now, by using the Lemma 2.4 and 2.5 we obtain the following result.

Theorem 3.8. Let $k \in (0,\infty)$ be fixed. For any continuous and non-decreasing $\varphi : [0,k] \to [0,k]$ satisfying $\varphi(x) \geq x$ for all $x \in [0,k]$ and any non-decreasing n-place function $H : [0,\infty)^n \to [0,\infty)$ such that H is continuous and bounded from below by maximum and any comonotone system $f_1, f_2, \ldots, f_n$ from $\mathcal{F}_k(X)$ and any fuzzy measure μ it holds

$$\begin{aligned}
&(S)\int_A H\left(\varphi\left(f_1\right), \varphi\left(f_2\right)\ldots, \varphi\left(f_n\right)\right)\mathrm{d}\mu\\
\leq\;& H\left(\varphi\left((S)\int_A f_1\,\mathrm{d}\mu\right), \varphi\left((S)\int_A f_2\,\mathrm{d}\mu\right)\ldots, \varphi\left((S)\int_A f_n\,\mathrm{d}\mu\right)\right).
\end{aligned} \tag{8}$$

Proof. Let $(S)\int_A H\left(\varphi\left(f_1\right), \varphi\left(f_2\right)\ldots, \varphi\left(f_n\right)\right)\mathrm{d}\mu = r \leq v < \infty$. Theorem 2.3(v) implies that:

$$\mu(A \cap \{x | H\left(\varphi\left(f_1\right), \varphi\left(f_2\right)\ldots, \varphi\left(f_n\right)\right) \geq r\}) \geq r.$$

Denote $A_1(\alpha) = \mu(A \cap \{x | f_1(x) \geq \alpha\})$, $A_2(\alpha) = \mu(A \cap \{x | f_2(x) \geq \alpha\}), \ldots, A_n(\alpha) = \mu(A \cap \{x | f_2(x) \geq \alpha\})$ and $C(\alpha) = \mu(A \cap \{x | H\left(\varphi\left(f_1\right), \varphi\left(f_2\right)\ldots, \varphi\left(f_n\right)\right) \geq \alpha)$. By Lemma 2.5 we have

$$(S)\int_A H\left(\varphi\left(f_1\right), \varphi\left(f_2\right)\ldots, \varphi\left(f_n\right)\right)\mathrm{d}\mu = (S)\int_0^v C(\alpha)m\left(\mathrm{d}\alpha\right).$$

Therefore, it is suffices to prove

$$(S)\int_0^v C(\alpha)m\left(\mathrm{d}\alpha\right) \leq H\left(\begin{array}{c} \varphi\left((S)\int_0^v A_1(\alpha)m\left(\mathrm{d}\alpha\right)\right), \varphi\left((S)\int_0^v A_2(\alpha)m\left(\mathrm{d}\alpha\right)\right), \\ \ldots, \varphi\left((S)\int_0^v A_n(\alpha)m\left(\mathrm{d}\alpha\right)\right)\end{array}\right).$$

Let $p_1 = (S)\int_0^v A_1(\alpha)m\,(\mathrm{d}\alpha)\,, p_2 = (S)\int_0^v A_2(\alpha)m\,(\mathrm{d}\alpha)\,, \ldots, p_n = (S)\int_0^v A_n(\alpha)m\,(\mathrm{d}\alpha)\,.$ Without loss of generality, let $p_1,\ p_2,\ldots,p_n < v$. Since $A_1(\alpha), A_2(\alpha),\ldots,A_n(\alpha)$ are non- increasing with respect to α, by Lemma 2.4 (moreover part), we have the following equalities:

$$A_1(p_1-) = p_1, A_2(p_2-) = p_2, \ldots, A_n(p_n-) = p_n.$$

Since $\varphi\,(x) \geq x$ for all $x \in [0,k]$ and $H : [0,\infty)^n \to [0,\infty)$ is continuous and bounded from below by maximum there holds

$$H\left(\varphi\left(p_1\right), \varphi\left(p_2\right), \ldots, \varphi\left(p_n\right)\right) \geq \max\left(\varphi\left(p_1\right), \varphi\left(p_2\right), \ldots, \varphi\left(p_n\right)\right) \geq \max\left(p_1, p_2 \ldots, p_n\right).$$

Now, on the contrary suppose

$$r > H\left(\varphi\left(p_1\right), \varphi\left(p_2\right), \ldots, \varphi\left(p_n\right)\right). \tag{9}$$

Then

$$\begin{aligned}
& \mu(A \cap \{x | H\left(\varphi\left(f_1\right), \varphi\left(f_2\right), \ldots, \varphi\left(f_n\right)\right) \geq r\}) \\
\leq\ & \mu\left(A \cap \{x | H\left(\varphi\left(f_1\right), \varphi\left(f_2\right), \ldots, \varphi\left(f_n\right)\right) > H\left(\varphi\left(p_1\right), \varphi\left(p_2\right), \ldots, \varphi\left(p_n\right)\right)\}\right) \\
\leq\ & \mu\left(A \cap \left(\begin{array}{c} \{x|\varphi\left(f_1\right)(x) > \varphi\left(p_1\right)\} \cup \{x|\varphi\left(f_2\right)(x) > \varphi\left(p_2\right)\} \\ \cup \ldots \cup \{x|\varphi\left(f_n\right)(x) > \varphi\left(p_n\right)\} \end{array}\right)\right) \\
\leq\ & \mu\left(A \cap \left(\{x|f_1\left(x\right) > p_1\} \cup \{x|f_2\left(x\right) > p_2\} \cup \ldots \cup \{x|f_n\left(x\right) > p_n\}\right)\right).
\end{aligned}$$

Therefore for sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned}
& \mu(A \cap \{x | H\left(\varphi\left(f_1\right), \varphi\left(f_2\right), \ldots, \varphi\left(f_n\right)\right) \geq r\}) \\
\leq\ & \mu\left(A \cap \left(\begin{array}{c} \{x|f_1\left(x\right) \geq p_1 - \varepsilon\} \cup \{x|f_2\left(x\right) \geq p_2 - \varepsilon\} \\ \cup \ldots \cup \{x|f_n\left(x\right) \geq p_n - \varepsilon\} \end{array}\right)\right).
\end{aligned}$$

Letting $\varepsilon \to 0$, then we have

$$\begin{aligned}
r\ &\leq\ \lim_{\varepsilon\to 0} \mu(A \cap \{x | H\left(\varphi\left(f_1\right), \varphi\left(f_2\right), \ldots, \varphi\left(f_n\right)\right) \geq r\}) \\
&\leq\ \lim_{\varepsilon\to 0} \left(\max(A_1(p_1-\varepsilon), A_2(p_2-\varepsilon), \ldots, A_n(p_n-\varepsilon))\right) \\
&=\ \max\left(p_1, p_2 \ldots, p_n\right) \leq H\left(\varphi\left(p_1\right), \varphi\left(p_2\right) \ldots, \varphi\left(p_n\right)\right),
\end{aligned}$$

which is a contradiction to (9). Hence $r \leq H\left(\varphi\left(p_1\right), \varphi\left(p_2\right), \ldots, \varphi\left(p_n\right)\right)$ and the proof is completed. □

Observe that if $H(0,..,0)$ for a function H required in Theorem 3.8 holds, then H is a disjunctive (continuous) aggregation function on $[0,k]$, see [7]. Typical examples in the case $k = 1$ of such aggregation functions are continuous t-conorms, cocopulas, etc.

Corollary 3.9. Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g \colon X \to [0,1]$ two comonotone measurable functions. If S is a continuous t-conorm and φ is continuous S -evaluator on X such that $\varphi\,(x) \geq x$, then the inequality

$$(S)\int_A S\left(\varphi\left(f\right), \varphi\left(g\right)\right) \mathrm{d}\mu \leq S\left(\varphi\left((S)\int_A f\,\mathrm{d}\mu\right), \varphi\left((S)\int_A g\,\mathrm{d}\mu\right)\right) \tag{10}$$

holds for any $A \in \mathcal{F}$.

The following example shows that $\varphi(x) \geq x$ for all $x \in [0, k]$ in Theorem 3.8 is inevitable.

Example 3.10. Let $X = [0,1]$, $f_1(x) = f_2(x) = x$, $\varphi(x) = x^2$ and $H(x,y) = \min\{1, x+y\}$ and the fuzzy measure μ be defined as $\mu(A) = m^2(A)$. Then

$$(S)\int_0^1 H(\varphi(f_1), \varphi(f_2))\, \mathrm{d}\mu = \bigvee_{\alpha\in[0,1]} \left(\alpha \wedge \left(1 - \frac{\sqrt{2\alpha}}{2}\right)\right) = 6 - 4\sqrt{2} = 0.343\,15$$

but

$$\varphi\left((S)\int_0^1 f_1\, \mathrm{d}\mu\right) = \varphi\left((S)\int_0^1 f_2\, \mathrm{d}\mu\right) = \left(\frac{3-\sqrt{5}}{2}\right)^2 = 0.145\,9.$$

Then

$$\begin{aligned}(S)\int_0^1 H(\varphi(f_1), \varphi(f_2))\, \mathrm{d}\mu &= 0.343\,15 > H\left(\varphi\left((S)\int_0^1 f_1\, \mathrm{d}\mu\right), \varphi\left((S)\int_0^1 f_2\, \mathrm{d}\mu\right)\right) \\ &= 0.291\,8,\end{aligned}$$

which violates Theorem 3.8.

The following example shows that the comonotonicity of $f_1, f_2, \ldots, f_n$ in Theorem 3.8 cannot be omitted.

Example 3.11. Let $X = [0,1]$, $f_1(x) = x$, $f_2(x) = (1-x)$, $H(x,y) = \min\{1, x+y\}$, $\varphi(x) = x$ and the fuzzy measure μ be defined as $\mu(A) = m^2(A)$. Then

$$(S)\int f_1\, \mathrm{d}\mu = (S)\int f_2\, \mathrm{d}\mu = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge (1-\alpha)^2\right] = 0.381\,97$$

and

$$(S)\int H(x, 1-x)\, \mathrm{d}\mu = 1.$$

Hence

$$\begin{aligned}(S)\int H(\varphi(f_1), \varphi(f_2))\, \mathrm{d}\mu &= 1 > H\left(\varphi\left((S)\int f_1\, \mathrm{d}\mu\right), \varphi\left((S)\int f_2\, \mathrm{d}\mu\right)\right) \\ &= 0.763\,94,\end{aligned}$$

which violates Theorem 3.8.

The following results are easy to obtain.

Corollary 3.12. Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g\colon X \to [0,1]$ two comonotone measurable functions. Let $\star\colon [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from below by maximum. Then

$$(S)\int_A f \star g \,\mathrm{d}\mu \leq \left((S)\int_A f \,\mathrm{d}\mu\right) \star \left((S)\int_A g \,\mathrm{d}\mu\right) \tag{11}$$

for any $A \in \mathcal{F}$.

Corollary 3.13. Let $f_1, f_2, \ldots, f_n$ be such that any two of them are comonotone and be as in Corollary 3.9. Then

$$\begin{aligned} &\left((S)\int_A \varphi\left(S\left(\ldots\left(S\left(S\left(f_1, f_2\right)\right), f_3\right)\ldots\right), f_n\right) \mathrm{d}\mu\right) \\ \leq\ & S\left(\begin{array}{c} S\left(\ldots\left(S\left(\varphi\left((S)\int_A f_1 \,\mathrm{d}\mu\right)\right), \left(\varphi\left((S)\int_A f_2 \,\mathrm{d}\mu\right)\right)\right), \ldots \\ \ldots, \left(\varphi\left((S)\int_A f_n \,\mathrm{d}\mu\right)\right) \end{array}\right). \end{aligned}$$

4. CONCLUSION

In this paper, we have investigated further development of Chebyshev type inequalities for Sugeno integrals and we have related them to T-evaluators and S-evaluators. As an interesting open problem for further investigation we pose the generalization of equality (3) for n-ary case. To be more precise, it is worth studying the case when the inequalities (5) and/or (8) became equalities, independently of incoming functions $f_1, \ldots, f_n$.

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Hamzeh Agahi, Department of Statistics, Faculty of Mathematics and Computer Science, Shahid Bahonar University of Kerman, Kerman, Iran
e-mail: Hamzeh.Agahi@gmail.com

Radko Mesiar, Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, SK-81368 Bratislava, Slovakia and Institute of Information Theory and Automation, Czech Academy of Sciences, CZ 182 08 Prague, Czech Republic
e-mail: mesiar@math.sk

Yao Ouyang, Faculty of Science, Huzhou Teacher's College, Huzhou, Zhejiang 313000, People's Republic of China
e-mail: oyy@hutc.zj.cn